Improving Braginskii theory for finite flow velocity

Jeong-Young Ji, J. Andrew Spencer, and Eric D. Held

Utah State University

CTTS Meeting

October 20 2019, Fort Lauderdale, FL

Kinetic equation and collision operator

- Kinetic equation for $f_a(t, \mathbf{r}, \mathbf{v})$

$$\frac{\partial f_a}{\partial t} + \mathbf{v} \cdot \frac{\partial f_a}{\partial \mathbf{r}} + \frac{q_a}{m_a}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f_a}{\partial \mathbf{v}} = \sum_b C(f_a, f_b)$$

- Landau collision operator
  - ▶ Landau (1937)

$$C(f_a, f_b) = \frac{\gamma_{ab}}{2m_a} \frac{\partial}{\partial \mathbf{v}} \cdot \int d\mathbf{v}' \mathsf{U} \cdot \left[\frac{\partial f_a(\mathbf{v})}{\partial \mathbf{v}} f_b(\mathbf{v}') - f_a(\mathbf{v}) \frac{m_a}{m_b} \frac{\partial f_b(\mathbf{v}')}{\partial \mathbf{v}'} \right]$$

where $\mathsf{U} = \dfrac{|\mathbf{v} - \mathbf{v}'|^2 \mathsf{I} - (\mathbf{v} - \mathbf{v}')(\mathbf{v} - \mathbf{v}')}{|\mathbf{v} - \mathbf{v}'|^3}$ and $\gamma_{ab} = \dfrac{q_a^2 q_b^2 \ln \Lambda_{ab}}{4\pi\epsilon_0^2 m_a} = \dfrac{3\sqrt{\pi} m_a v_{Ta}^3}{4 n_b \tau_{ab}}$

  - ▶ Rosenbluth, MacDonald, and Judd (1957)

$$C(f_a, f_b) = \frac{\gamma_{ab}}{2m_a} \frac{\partial}{\partial \mathbf{v}} \cdot \left[\frac{\partial}{\partial \mathbf{v}} \cdot \left(f_a \frac{\partial}{\partial \mathbf{v}} \frac{\partial G_b^+}{\partial \mathbf{v}} \right) - 2\left(1 + \frac{m_a}{m_b}\right) f_a \frac{\partial G_b^-}{\partial \mathbf{v}} \right]$$

where

$$G_b^{\pm}(\mathbf{v}) = \int d\mathbf{v}' |\mathbf{v} - \mathbf{v}'|^{\pm 1} f_b(\mathbf{v}')$$

General moment expansion [Ji and Held 2006 2008 2009]
Irreducible Hermite polynomials [Chapman 1916 Enskog 1917 Grad 1963]

$$\mathbf{c}_a = \frac{\mathbf{w}_a}{v_{Ta}} = \frac{\mathbf{v} - \mathbf{V}_a}{v_{Ta}}, \quad v_{Ta} = \sqrt{\frac{2T_a}{m_a}}, \quad \text{vs.} \quad \mathbf{s}_a = \frac{\mathbf{v}}{v_{Ta}}$$

- Moment expansion: m_a^{lk} (vs. M_a^{lk}) are symmetric traceless fluid moments

$$f_a(t,\mathbf{r},\mathbf{v}) = f_a^{\mathrm{M}} \sum_{lk} \mathsf{m}_a^{lk}(t,\mathbf{r}) \cdot \hat{\mathsf{p}}_a^{lk} \quad \text{vs.} \quad f_a(t,\mathbf{r},\mathbf{v}) = f_a^{\mathrm{m}} \sum_{lk} \mathsf{M}_a^{lk}(t,\mathbf{r}) \cdot \hat{\mathsf{P}}_a^{lk}$$

$$\mathsf{n}_a^{lk}(t,\mathbf{r}) \equiv n_a \mathsf{m}_a^{lk} = \int d\mathbf{v} \hat{\mathsf{p}}_a^{lk} f_a \qquad \mathsf{N}_a^{lk}(t,\mathbf{x}) \equiv n_a \mathsf{M}_a^{lk} = \int d\mathbf{v} \hat{\mathsf{P}}_a^{lk} f_a$$

- $\hat{\mathsf{p}}^{lk}$'s are orthonormal, irreducible, tensorial polynomials and form a complete set

$$\int d\mathbf{v} \hat{\mathsf{p}}^{jp} \hat{\mathsf{p}}^{lk} \cdot \mathsf{m}^{lk} f^{\mathrm{M}} = \delta_{jl}\delta_{pk} \mathsf{n}^{jp}, \quad \hat{\mathsf{p}}^{lk} = \frac{1}{\sqrt{\sigma_{lk}}} \mathsf{p}^{lk}$$

$$\begin{aligned} \mathsf{p}_a^{lk} &= \mathsf{P}^l(\mathbf{c}_a) L_k^{(l+1/2)}(c_a^2) \quad \text{vs.} \quad \mathsf{P}_a^{lk} = \mathsf{P}^l(\mathbf{s}_a) L_k^{(l+1/2)}(s_a^2) \\ &= \text{(harmonic tensor)(associated Laguerre polynomial)} \end{aligned}$$

$$f_a^{\mathrm{M}} = \frac{n_a}{\pi^{3/2} v_{Ta}^3} e^{-c_a^2} \text{ (Maxwellian distribution)} \quad \text{vs.} \quad f_a^{\mathrm{m}} = \frac{n_a}{\pi^{3/2} v_{Ta}^3} e^{-s_a^2}$$

Several low order moments (21 moments)

$p^{lk} = P^l(\mathbf{c}) L_k^l(c^2)$	$L_k^l = L_k^{(l+\frac{1}{2})}$	n^{lk}	fluid moment equation	indep.
$P^0 = 1$	$L_0^0 = 1$	n	density (n)	1
	$L_1^0 = \frac{3}{2} - c^2$	0	temperature (T)	1
$P^1 = \mathbf{c}$	$L_0^1 = 1$	0	flow velocity $(\mathbf{V})$	3
	$L_1^1 = \frac{5}{2} - c^2$	n^{11}	heat flow $(\mathbf{h})$	3
	$L_2^1 = \frac{35}{8} - \frac{7}{2}c^2 + \frac{1}{2}c^4$	n^{12}	heat w. heat flow $(\mathbf{r})$	3
$P^2 = \mathbf{cc} - \frac{c^2}{3}\mathsf{I}$	$L_0^2 = 1$	n^{20}	viscosity $(\boldsymbol{\pi})$	5
	$L_1^2 = \frac{7}{2} - c^2$	n^{21}	heat viscosity $(\boldsymbol{\theta})$	5

$f = f^{\mathrm{M}\{n,\mathbf{V},T\}}\Big(1 + \cdots$ ← scalar moments

$+ \sigma_{\mathbf{h}} p^{11} \cdot \mathbf{h} + \sigma_{\mathbf{r}} p^{12} \cdot \mathbf{r} + \cdots$ ← vector moments

$+ \sigma_{\boldsymbol{\pi}} p^{20} \cdot \boldsymbol{\pi} + \sigma_{\boldsymbol{\theta}} p^{21} \cdot \boldsymbol{\theta} + \cdots$ ← rank-2 tensor moments

$+ \cdots \Big)$ ← higher rank tensor momemnts

Fluid equations and closures

Maxwellian moment $(n_a, \mathbf{V}_a, T_a)$ equations

(0,0) $d_t n_a + n_a \nabla\cdot\mathbf{V}_a = 0\,(d_t \equiv \partial_t + \mathbf{V}_a\cdot\nabla)$

(0,1) $\frac{3}{2} n_a d_t T_a + n_a T_a \nabla\cdot\mathbf{V}_a + \nabla\cdot\mathbf{h}_a + \nabla\mathbf{V}_a : \boldsymbol{\pi}_a = Q_a$

(1,0) $m_a n_a d_t \mathbf{V}_a - n_a q_a(\mathbf{E} + \mathbf{V}_a\times\mathbf{B}) + \nabla p_a + \nabla\cdot\boldsymbol{\pi}_a = \mathbf{R}_a$

General moment equations $D\mathsf{n} + \Omega\mathbf{b}\breve{\times}\mathsf{n} = C\mathsf{n}$ ($\mathsf{n}^{lk} \to v^{l+2k}$ moment)

$$
\begin{aligned}
(1,1)\ & d_t\mathbf{h} + \Omega\mathbf{b}\times\mathbf{h} + \tfrac{7}{5}(\nabla\cdot\mathbf{V})\mathbf{h} + \tfrac{7}{5}\mathbf{h}\cdot(\nabla\mathbf{V}) + \tfrac{2}{5}(\nabla\mathbf{V})\cdot\mathbf{h} + \tfrac{5p}{2m}\nabla T \\
& + \tfrac{T}{m}\nabla\cdot\boldsymbol{\pi} + \tfrac{7}{2}\tfrac{\nabla T}{m}\cdot\boldsymbol{\pi} - \mathbf{a}\cdot\boldsymbol{\pi} + \nabla\cdot\boldsymbol{\theta} + \tfrac{1}{3}\nabla\mathsf{u}^{02} + \nabla\mathbf{V} : \mathsf{u}^{30} \\
& = C^1_{10}\mathbf{V}_{\mathrm{ei}} + C^1_{11}\mathbf{h} + C^1_{12}\mathbf{r} + \cdots\ (\mathbf{h}\ \text{heat flow}) \\
(1,2)\ & d_t\mathbf{r} + \Omega\mathbf{b}\times\mathbf{r} + \cdots = C^1_{10}\mathbf{V}_{\mathrm{ei}} + C^1_{21}\mathbf{h} + C^1_{22}\mathbf{r} + \cdots \quad (\mathbf{r}\ \text{heat heat flow}) \\
(2,0)\ & d_t\boldsymbol{\pi} + \Omega\mathbf{b}\breve{\times}\boldsymbol{\pi} + (\nabla\cdot\mathbf{V})\boldsymbol{\pi} + 2\overline{\boldsymbol{\pi}\cdot(\nabla\mathbf{V})} + p\mathsf{W} + \tfrac{4}{5}\overline{\nabla\mathbf{h}} + \nabla\cdot\mathsf{u}^{30} \\
& = C^2_{00}\boldsymbol{\pi} + C^2_{01}\boldsymbol{\theta} + \cdots\ (\boldsymbol{\pi}\ \text{viscosity}) \\
(2,1)\ & d_t\boldsymbol{\theta} + \Omega\mathbf{b}\breve{\times}\boldsymbol{\pi} + \cdots = C^2_{10}\boldsymbol{\pi} + C^2_{11}\boldsymbol{\theta} + \cdots\ (\boldsymbol{\theta}\ \text{heat viscosity}) \\
& \text{where } \mathbf{a} = \tfrac{q}{m}(\mathbf{E} + \mathbf{V}\times\mathbf{B}) - d_t\mathbf{V} \text{ and } \mathsf{W} = \nabla\mathbf{V} + (\nabla\mathbf{V})^{\mathrm{T}} - \tfrac{2}{3}\nabla\cdot\mathbf{V}\mathsf{I}
\end{aligned}
$$

Closures: express $\mathbf{h}_a(\mathsf{n}^{11}_a)$, $\boldsymbol{\pi}_a(\mathsf{n}^{20}_a)$, Q_a, $\mathbf{R}_a$ in terms of $n_a, \mathbf{V}_a, T_a$

$$\mathbf{h}_{\mathrm{e}} = -\kappa^{\mathrm{e}}_{\|}\nabla_{\|}T_{\mathrm{e}} - \kappa^{\mathrm{e}}_{\perp}\nabla_{\perp}T_{\mathrm{e}} - \kappa^{\mathrm{e}}_{\times}\nabla_{\times}T_{\mathrm{e}} + \beta_{\|}T_{\mathrm{e}}\mathbf{V}_{\mathrm{ei}\|} + \beta_{\perp}T_{\mathrm{e}}\mathbf{V}_{\mathrm{ei}\perp} + \beta_{\times}T_{\mathrm{e}}\mathbf{V}_{\mathrm{ei}\times}$$

$$\mathbf{R}_{\mathrm{e}} = -\alpha_{\|}\mathbf{V}_{\mathrm{ei}\|} - \alpha_{\perp}\mathbf{V}_{\mathrm{ei}\perp} + \alpha_{\times}\mathbf{V}_{\mathrm{ei}\times} - \beta_{\|}\nabla_{\|}T_{\mathrm{e}} - \beta_{\perp}\nabla_{\perp}T_{\mathrm{e}} - \beta_{\times}\nabla_{\times}T_{\mathrm{e}}$$

Moments of the Landau collision operator

$$\frac{\gamma_{ab}}{2m_a}\int d\mathbf{v}\mathsf{P}^{jp}(\mathbf{c}_a)\frac{\partial}{\partial\mathbf{v}}\cdot\left[\frac{\partial}{\partial\mathbf{v}}\cdot\left(f_a\frac{\partial^2 G_{+,b}}{\partial\mathbf{v}\partial\mathbf{v}}\right)-2\left(1+\frac{m_a}{m_b}\right)\frac{\partial}{\partial\mathbf{v}}\cdot\left(f_a\frac{\partial G_{-,b}}{\partial\mathbf{v}}\right)\right]$$

Rosenbluth potential $G_{\pm,b}(\mathbf{v})=\int d\mathbf{v}' f_b(\mathbf{v}')|\mathbf{v}-\mathbf{v}'|^{\pm 1}$

- Exact collisional moments in the total velocity expansion [Ji and Held 2006]

$$\int d\mathbf{v}\mathsf{P}^{jp}(\mathbf{s}_a)C\left(f_a^{\mathrm{M}}\mathsf{M}_a^{lk}\cdot\mathsf{P}^{lk}(\mathbf{s}_a),f_b^{\mathrm{M}}\right) = \delta_{jl}\sigma_j A_{ab}^{jpk}\mathsf{M}_a^{lk}$$

$$\int d\mathbf{v}\mathsf{P}^{jp}(\mathbf{s}_a)C\left(f_a^{\mathrm{M}},f_b^{\mathrm{M}}\mathsf{M}_b^{lk}\cdot\mathsf{P}^{lk}(\mathbf{s}_b)\right) = \delta_{jl}\sigma_j B_{ab}^{jpk}\mathsf{M}_b^{lk}$$

- Small mass ratio approximation in the random velocity expansion [Ji and Held 2009]

$$\int d\mathbf{v}\mathsf{P}^{jp}(\mathbf{c}_\mathrm{e})C\left(f_\mathrm{e}^{\mathrm{M}}\mathsf{m}_\mathrm{e}^{lk}\cdot\mathsf{P}^{lk}(\mathbf{c}_\mathrm{e}),f_\mathrm{i}^{\mathrm{M}}\right) = \sigma_j A_{\mathrm{ei}}^{jp,lk}\mathsf{m}_\mathrm{e}^{lk}$$

$$\int d\mathbf{v}\mathsf{P}^{jp}(\mathbf{c}_\mathrm{i})C\left(f_\mathrm{e}^{\mathrm{M}},f_\mathrm{i}^{\mathrm{M}}\mathsf{m}_\mathrm{i}^{nq}\cdot\mathsf{P}^{nq}(\mathbf{c}_\mathrm{i})\right) = \sigma_j B_{\mathrm{ei}}^{jp,lk}\mathsf{m}_\mathrm{i}^{lk}$$

  * Assumed $v_{T\mathrm{i}}\ll v_{T\mathrm{e}}$ and $|\mathbf{V}_\mathrm{e}-\mathbf{V}_\mathrm{i}|\ll v_{T\mathrm{e}}$: not valid for runaway electrons
  * Need to calculate the collisional moments for arbitrary relative flow velocity

Exact collisional moments for arbitrary relative flow velocity: Calculation steps

- Change derivatives, $\dfrac{\partial G_\pm(\mathbf{v}-\mathbf{V}_b)}{\partial \mathbf{v}} = -\dfrac{\partial G_\pm}{\partial \mathbf{V}_b}$
- Integrate by parts to differentiate each term of polynomial $\mathsf{P}^{jp}(\mathbf{c}_a)$
- Change velocity variables [Schunk and Nagy 2000]

$$\begin{array}{c} \mathbf{w}_\star = X_{ba}\mathbf{w}_a + X_{ab}\mathbf{w}'_b \\ \mathbf{w}_\dagger = \mathbf{w}_a - \mathbf{w}'_b \end{array} \leftrightarrow \begin{array}{c} \mathbf{w}_a = v_\star(\mathbf{c}_\star + \chi_a \mathbf{c}_\dagger) \\ \mathbf{w}'_b = v_\star(\mathbf{c}_\star + \chi'_b \mathbf{c}_\dagger) \end{array}, \qquad \begin{array}{c} v_\star^{-2} = v_{Ta}^{-2} + v_{Tb}^{-2} \\ v_\dagger^2 = v_{Ta}^2 + v_{Tb}^2 \end{array}$$

where $X_{ab} = (1 + v_{Ta}^2/v_{Tb}^2)^{-1}$, $X_{ba} = 1 - X_{ab}$, $\chi_a = v_{Ta}/v_{Tb}$, $\chi'_b = -v_{Tb}/v_{Ta}$

$$\int d\mathbf{v} \int d\mathbf{v}' f_a^{\mathrm{M}} f_b^{\mathrm{M}} |\mathbf{v}-\mathbf{v}'|^{\pm} = n_a n_b v_\dagger^{\pm 1} \int d\mathbf{c}_\star \int d\mathbf{c}_\dagger \frac{e^{-c_\star^2}}{\pi^{3/2}} \frac{e^{-c_\dagger^2}}{\pi^{3/2}} |\mathbf{c}_\dagger - \mathbf{x}|^{\pm}$$

where $\mathbf{x} = (\mathbf{V}_b - \mathbf{V}_a)/v_\dagger$

- Expand all $\mathbf{c}_a$ and $\mathbf{c}_b$ variables: involve multiple summations
- Perform $\mathbf{c}_\star$ integration using $\displaystyle\int d\mathbf{c}_\star \mathbf{c}_\star^n c^{2u} \frac{e^{-c_\star^2}}{\pi^{3/2}} = \frac{2[u+(n+1)/2]!}{\pi^{1/2}(n+1)}\{\mathsf{I}^{n/2}\}$
- Simplify all inner-products and symmetrizations
- Perform $\mathbf{c}_\dagger$ integration using the Rosenbluth potentials [Ji and Held 2006]
- Differentiate with respect to $\mathbf{V}_b$ ($\mathbf{x}$)
- Final results involve only algebraic summations of $G_\pm^{np}(x)\left\{\mathbf{x}^j \cdot^r \mathsf{m}^{lk}\mathsf{I}^i\right\}$ terms

Examples: vector moments due to relative flow

$$
\begin{aligned}
\mathsf{A}_{ab}^{10,00} &= \frac{3\sqrt{\pi}(1+\mu)X}{4\tau_{ab}\mu}\frac{E-xE'}{x^2}\hat{\mathbf{x}}_{ba}n_a \\
\mathsf{A}_{ab}^{11,00} &= \frac{9\sqrt{\pi}X}{4\tau_{ab}\mu}\frac{\left[\mu x+(1+\mu)x^3X\right]E'-\mu E}{x^2}\hat{\mathbf{x}}_{ba}n_a \\
\mathsf{A}_{ab}^{20,00} &= \frac{3\sqrt{\pi}\sqrt{X}}{8\tau_{ab}\mu}\left[\left(\frac{3\mu+6X(1+\mu)}{x^3}-\frac{2\mu}{x}\right)E\right. \\
&\qquad \left.-\left(\frac{3\mu+6X(1+\mu)}{x^2}+4X(1+\mu)\right)E'\right]\overline{\hat{\mathbf{x}}_{ba}\hat{\mathbf{x}}_{ba}}
\end{aligned}
$$

Small mass ratio approximation for $ab=\mathrm{ei}$:
$\mu^{-1}=m_\mathrm{e}/m_\mathrm{i}\ll 1$, $X=(1+v_{T\mathrm{i}}^2/v_{T\mathrm{e}}^2)^{-1}\to 1$, $x\to|\mathbf{V}_\mathrm{i}-\mathbf{V}_\mathrm{e}|/v_{T\mathrm{e}}$

$$
\begin{aligned}
\mathsf{A}_{\mathrm{ei}}^{10,00} &\approx \frac{3\sqrt{\pi}}{4\tau_\mathrm{ei}}\frac{E-xE'}{x^2}\hat{\mathbf{x}}_\mathrm{ie}n_\mathrm{e} \xrightarrow{x\ll 1} \frac{1}{\tau_\mathrm{ei}}x\hat{\mathbf{x}}_\mathrm{ie}n_\mathrm{e} \\
\mathsf{A}_{\mathrm{ei}}^{11,00} &\approx \frac{9\sqrt{\pi}}{4\tau_\mathrm{ei}}\frac{x(x^2+1)E'-E}{x^2}\hat{\mathbf{x}}_\mathrm{ie}n_\mathrm{e} \xrightarrow{x\ll 1} \frac{3}{2\tau_\mathrm{ei}}x\hat{\mathbf{x}}_\mathrm{ie}n_\mathrm{e} \\
\mathsf{A}_{\mathrm{ei}}^{20,00} &\approx \frac{3\sqrt{\pi}}{8\tau_\mathrm{ei}}\left[\left(\frac{9}{x^3}-\frac{2}{x}\right)E-\left(\frac{9}{x^2}+4\right)E'\right]\overline{\hat{\mathbf{x}}_\mathrm{ie}\hat{\mathbf{x}}_\mathrm{ie}}n_\mathrm{e} \xrightarrow{x\ll 1} \mathcal{O}(x^2)
\end{aligned}
$$

Collision matrix element: Heat flux - heat flux

$$
\begin{aligned}
\mathsf{A}_{ab}^{11,11} &= \frac{3\sqrt{\pi}X^{3/2}}{32\mu\tau_{ab}}\Bigg[\mathsf{n}_a^{11}\left(A_{0E}^{11,11}\frac{E}{x^3}+A_{0e}^{11,11}\frac{E'}{x^2}\right) \\
&\quad +\hat{\mathbf{x}}\hat{\mathbf{x}}\cdot\mathsf{n}_a^{11}\left(A_{1E}^{11,11}\frac{E}{x^3}+A_{1e}^{11,11}\frac{E'}{x^2}\right)\Bigg] \\
\mathsf{B}_{ab}^{11,11} &= \frac{3\sqrt{\pi}X^{3/2}\chi}{32\mu\tau_{ab}}\left(\mathsf{n}_a^{11}B_{0e}^{11,11}+\hat{\mathbf{x}}\hat{\mathbf{x}}\cdot\mathsf{n}_a^{11}B_{1e}^{11,11}\right)E'
\end{aligned}
$$

where $\hat{\mathbf{x}}=\hat{\mathbf{x}}_{ba}$ and $\chi=\chi_{ab}$ and

$$
\begin{aligned}
A_{0E}^{11,11} &= -2(9+17\mu) \\
A_{0e}^{11,11} &= (18+34\mu)-2x^2\left(4+6\mu+15X^2(1+\mu)-2X(11+14\mu)\right) \\
&\quad +4x^4X(-2+3X)(1+\mu) \\
A_{1E}^{11,11} &= 6(9+17\mu) \\
A_{1e}^{11,11} &= -6(9+17\mu)-4x^2(9+17\mu)+4x^4X(-16-22\mu+21X(1+\mu)) \\
&\quad -24x^6X^2(1+\mu) \\
B_{0e}^{11,11} &= 6(-1+X)(2\mu-5X(1+\mu))+12x^2(-1+X)X(1+\mu) \\
B_{1e}^{11,11} &= -12x^2(-1+X)(2\mu-7X(1+\mu))-24x^4(-1+X)X(1+\mu)
\end{aligned}
$$

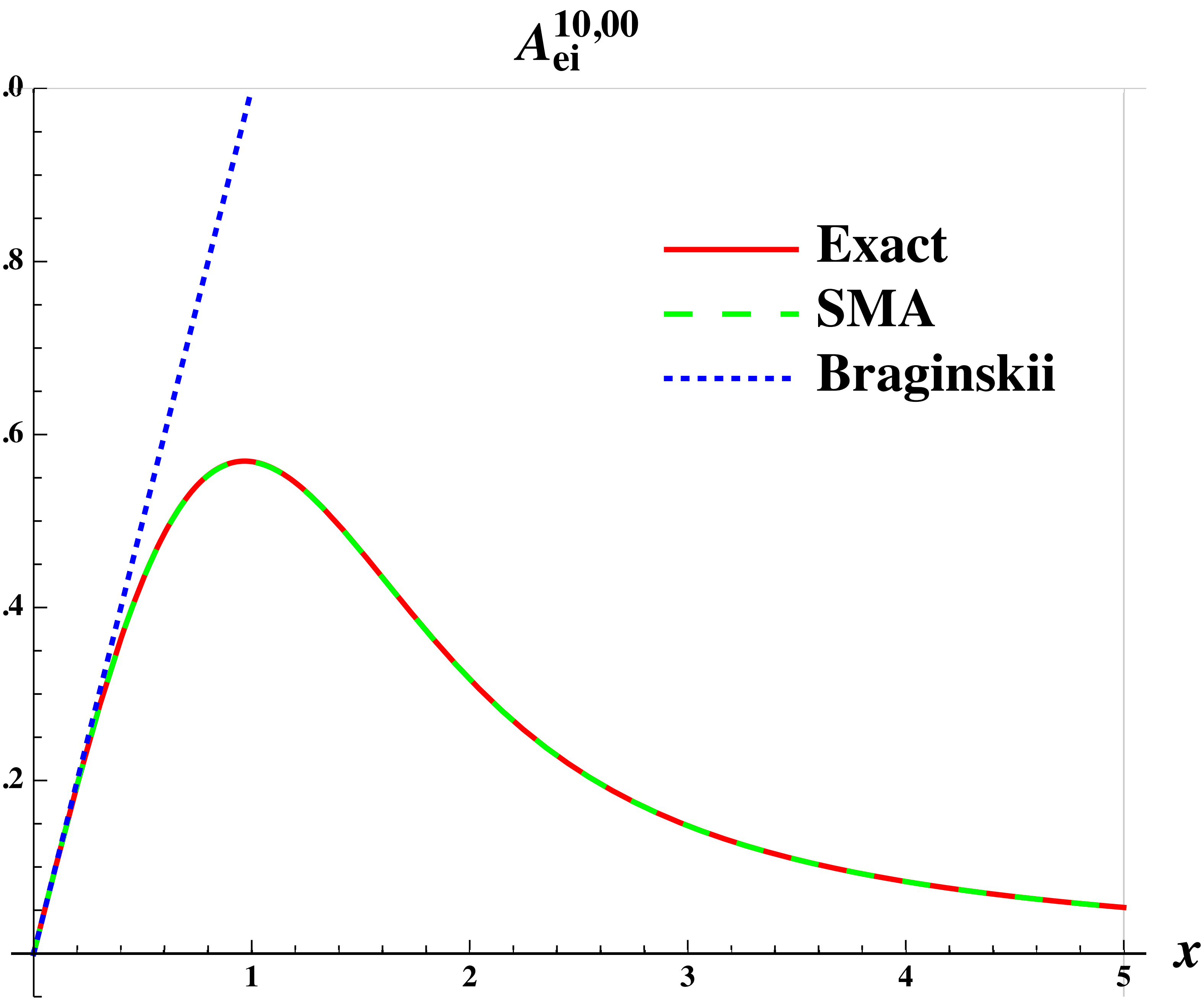
Collision matrix element: Viscosity - viscosity

$$
\begin{aligned}
\mathsf{A}_{ab}^{20,20} &= \frac{3\sqrt{\pi}X^{3/2}}{32\mu\tau_{ab}}\Bigg[\mathsf{n}_a^{20}\left(A_{0E}^{20,20}\frac{E}{x^5}+A_{0e}^{20,20}\frac{E'}{x^4}\right)\\
&+\overline{\hat{\mathbf{x}}\hat{\mathbf{x}}\cdot\mathsf{n}_a^{20}}4\left(A_{1E}^{20,20}\frac{E}{x^5}+A_{1e}^{200}\frac{E'}{x^4}\right)\\
&+\overline{\hat{\mathbf{x}}\hat{\mathbf{x}}}\hat{\mathbf{x}}\hat{\mathbf{x}}:\mathsf{n}_a^{20}\left(A_{2E}^{20,20}\frac{E}{x^5}+A_{2e}^{20,20}\frac{E'}{x^4}\right)\Bigg]
\end{aligned}
$$

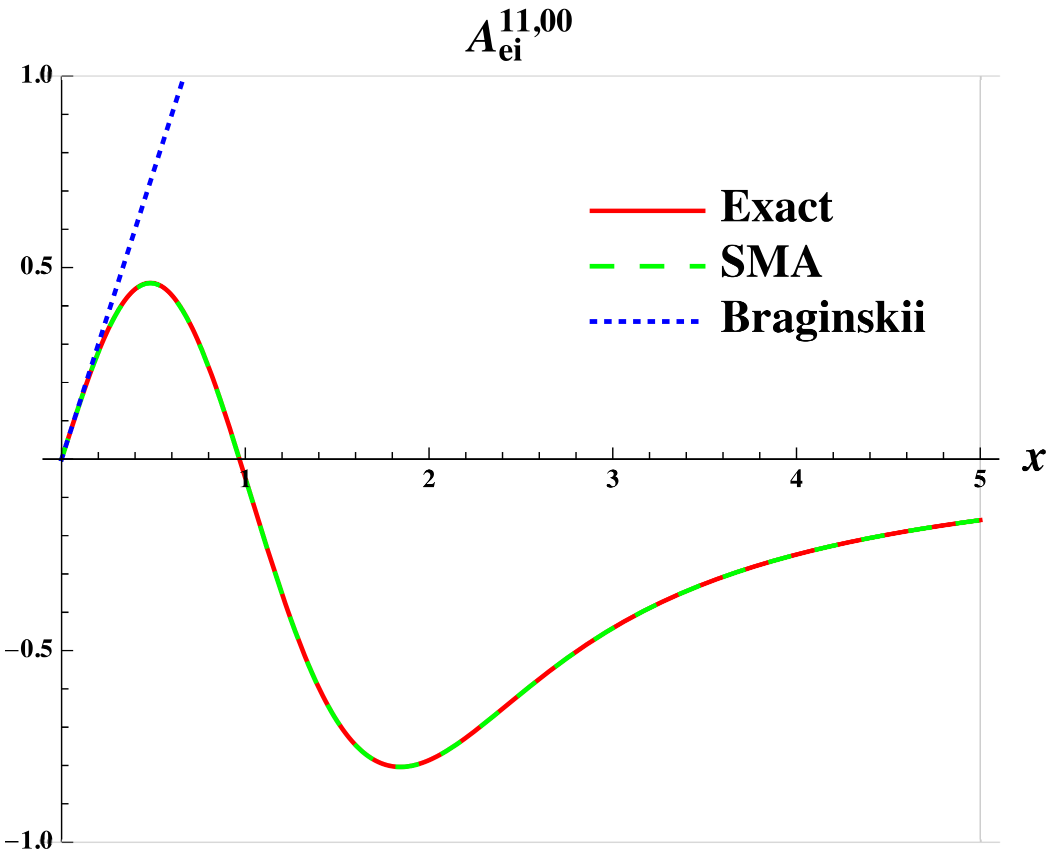
where

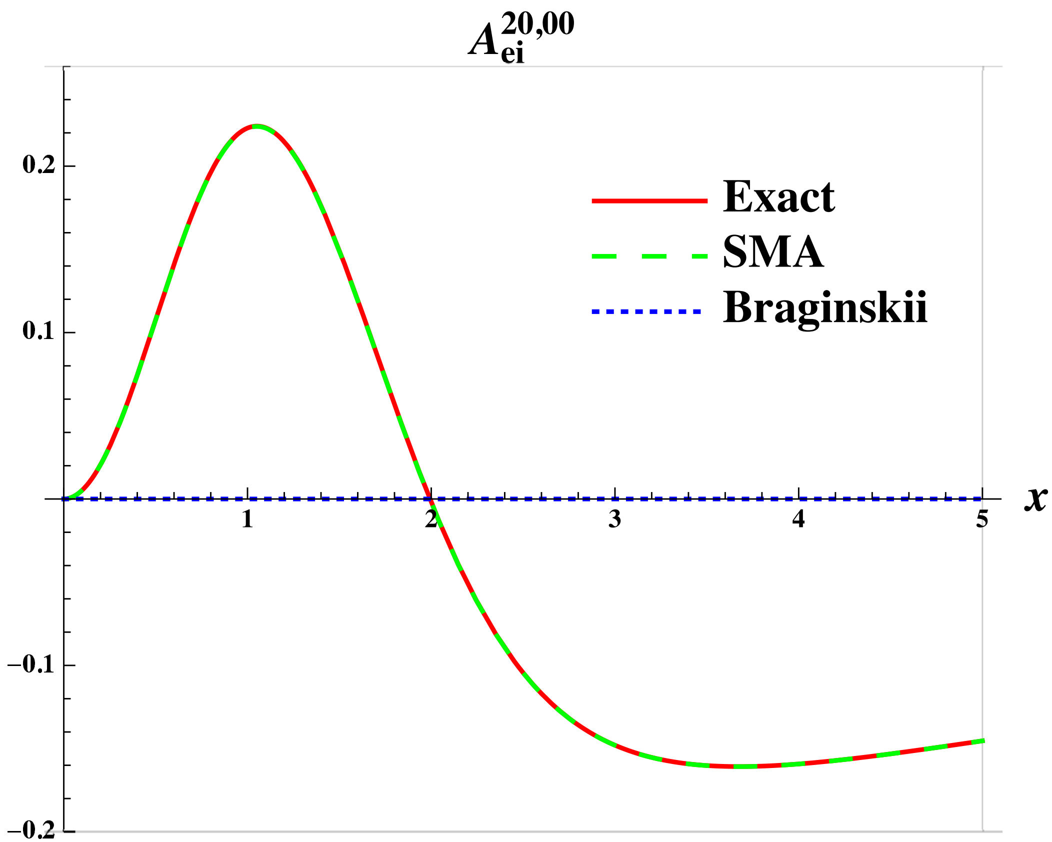
$$
\begin{aligned}
A_{0E}^{20,20} &= 6(\mu+2X(1+\mu))-4x^2(2+3\mu)\\
A_{0e}^{20,20} &= (-6\mu-12X(1+\mu))+8x^2X(1+\mu)\chi^2\\
A_{1E}^{20,20} &= -15(\mu+2X(1+\mu))+6x^2(1+2\mu)\\
A_{1e}^{20,20} &= (15\mu+30X(1+\mu))+x^2(4\mu+2(-3+10X)(1+\mu))\\
&\quad +2x^4(-2+4X)(1+\mu)\\
A_{2E}^{20,20} &= 15(7\mu+14X(1+\mu))-30x^2\mu\\
A_{2e}^{20,20} &= (-105\mu-210X(1+\mu))+x^2(-40\mu-140X(1+\mu))\\
&\quad +x^4(-8\mu-56X(1+\mu))-16x^6X(1+\mu)
\end{aligned}
$$

Momentum $(1,0)$ moment due to relative flow: Dreicer's friction (1959)



Heat flux $(1,1)$ and viscosity $(2,0)$ moments due to relative flow

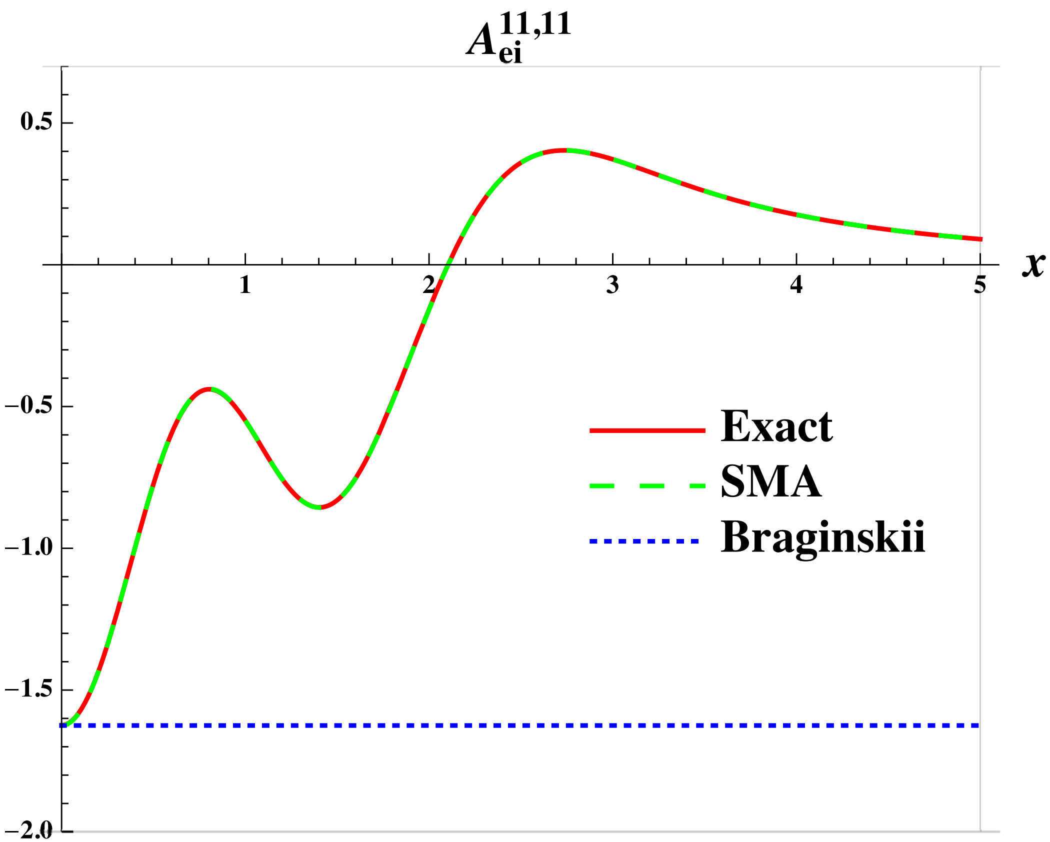


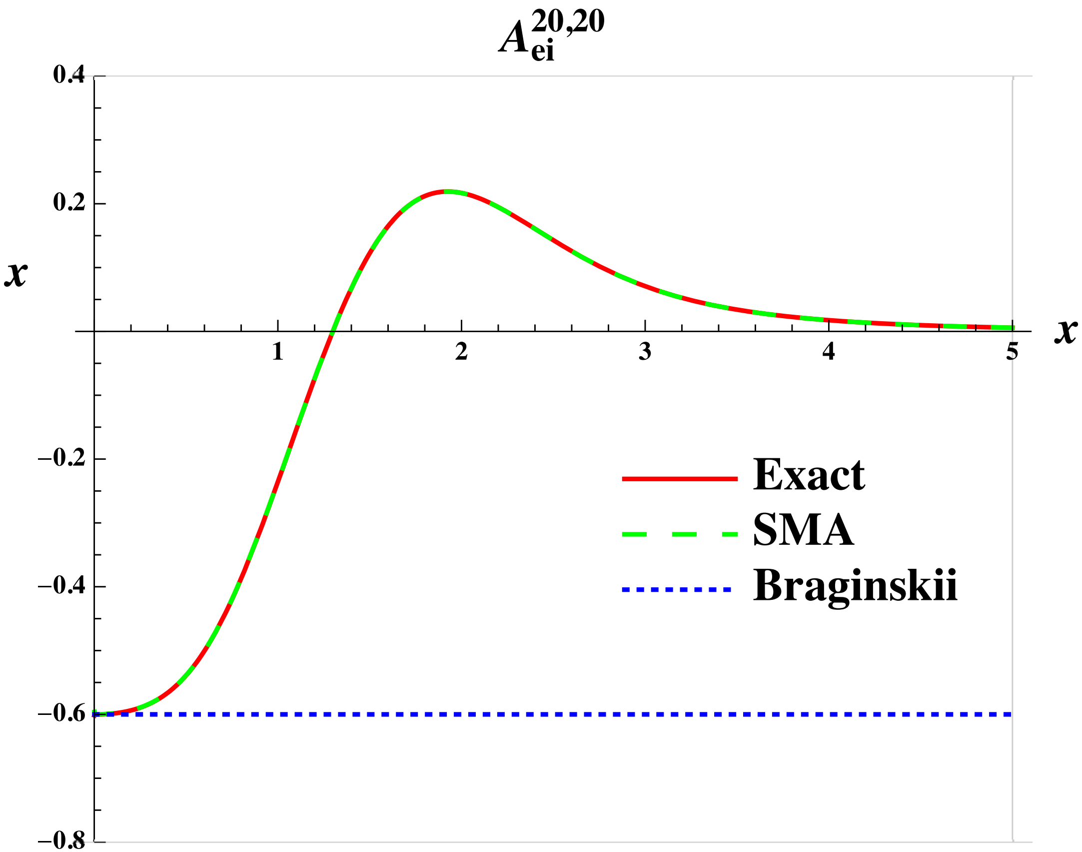


- Vector drives are proportional to x for $x \ll 1$ in Braginskii theory $\rightarrow \mathbf{h}_{\mathrm{e}} = -\kappa_{\mathrm{e}}(x)\nabla T_{\mathrm{e}} + \beta(x) T \mathbf{V}_{\mathrm{ei}}$ (where $\mathbf{V}_{ab} = \mathbf{V}_a - \mathbf{V}_b$)
- Viscosity drive not considered in Braginskii ($\boldsymbol{\pi} = 2\eta\overline{\nabla\mathbf{V}}$)

$$\overline{\mathbf{xx}} = \mathbf{xx} - \frac{1}{3}x^2\mathsf{I} \text{ where } \mathbf{x} = \frac{\mathbf{V}_{\mathrm{ie}}}{v_{T\mathrm{e}}} \rightarrow \boldsymbol{\pi} = 2\eta(x)\overline{\nabla\mathbf{V}} + \gamma(x)\overline{\mathbf{V}_{\mathrm{ie}}\mathbf{V}_{\mathrm{ie}}} + \lambda(x)\overline{\mathbf{V}_{\mathrm{ie}}\nabla T}$$

Collision matrix elements: Heat flux - heat flux and viscosity - viscosity





- Electron collision operator $\hat{C}_{\rm e} = \hat{C}_{\rm ee} + Z\hat{A}_{\rm ei}$

$$\hat{C}_{\rm ee}^{11,11} = -\frac{2\sqrt{2}}{5} \approx -0.566$$

$$\hat{C}_{\rm ee}^{20,20} = -\frac{3\sqrt{2}}{5} \approx -0.849$$

Summary and future work

- Formulas are implemented in `Mathematica` and verified against existing results
- Obtain closures in high collisionality
  - Check convergence with increasing the number of moments
- Investigate the behavior of closures low collisionality regimes
  - Parallel integral closures
- Consider magnetic field effects
  - Inhomogeneous along the field line
  - Curvature effects (centrifugal force)
  - Modify perpendicular closures