Fluid closures for runaway electrons (preliminary work)

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Fluid equations and closures

Maxwellian moment $(n_a, \mathbf{V}_a, T_a)$ equations

(0,0) $d_t n_a + n_a \nabla\cdot\mathbf{V}_a = 0 \, (d_t \equiv \partial_t + \mathbf{V}_a\cdot\nabla)$

(0,1) $\frac{3}{2} n_a d_t T_a + n_a T_a \nabla\cdot\mathbf{V}_a + \nabla\cdot\mathbf{h}_a + \nabla\mathbf{V}_a : \boldsymbol{\pi}_a = Q_a$

(1,0) $m_a n_a d_t \mathbf{V}_a - n_a q_a(\mathbf{E} + \mathbf{V}_a\times\mathbf{B}) + \nabla p_a + \nabla\cdot\boldsymbol{\pi}_a = \mathbf{R}_a$

General moment equations $D\mathsf{n} + \Omega\mathbf{b}\breve{\times}\mathsf{n} = C\mathsf{n}$ ($\mathsf{n}^{lk} \to v^{l+2k}$ moment)

$$
\begin{aligned}
&(1,1)\ d_t\mathbf{h} + \Omega\mathbf{b}\times\mathbf{h} + \tfrac{7}{5}(\nabla\cdot\mathbf{V})\mathbf{h} + \tfrac{7}{5}\mathbf{h}\cdot(\nabla\mathbf{V}) + \tfrac{2}{5}(\nabla\mathbf{V})\cdot\mathbf{h} + \frac{5p}{2m}\nabla T \\
&\qquad + \frac{T}{m}\nabla\cdot\boldsymbol{\pi} + \frac{7}{2}\frac{\nabla T}{m}\cdot\boldsymbol{\pi} - \mathbf{a}\cdot\boldsymbol{\pi} + \nabla\cdot\boldsymbol{\theta} + \tfrac{1}{3}\nabla\mathsf{u}^{02} + \nabla\mathbf{V} : \mathsf{u}^{30} \\
&\qquad = C^1_{10}\mathbf{V}_{\mathrm{ei}} + C^1_{11}\mathbf{h} + C^1_{12}\mathbf{r} + \cdots\ (\mathbf{h}\ \text{heat flow}) \\
&(1,2)\ d_t\mathbf{r} + \Omega\mathbf{b}\times\mathbf{r} + \cdots = C^1_{10}\mathbf{V}_{\mathrm{ei}} + C^1_{21}\mathbf{h} + C^1_{22}\mathbf{r} + \cdots \quad (\mathbf{r}\ \text{heat heat flow}) \\
&(2,0)\ d_t\boldsymbol{\pi} + \Omega\mathbf{b}\breve{\times}\boldsymbol{\pi} + (\nabla\cdot\mathbf{V})\boldsymbol{\pi} + 2\overline{\boldsymbol{\pi}\cdot(\nabla\mathbf{V})} + p\mathsf{W} + \tfrac{4}{5}\overline{\nabla\mathbf{h}} + \nabla\cdot\mathsf{u}^{30} \\
&\qquad = C^2_{00}\boldsymbol{\pi} + C^2_{01}\boldsymbol{\theta} + \cdots\ (\boldsymbol{\pi}\ \text{viscosity}) \\
&(2,1)\ d_t\boldsymbol{\theta} + \Omega\mathbf{b}\breve{\times}\boldsymbol{\pi} + \cdots = C^2_{10}\boldsymbol{\pi} + C^2_{11}\boldsymbol{\theta} + \cdots\ (\boldsymbol{\theta}\ \text{heat viscosity})
\end{aligned}
$$

where $\mathbf{a} = \frac{q}{m}(\mathbf{E} + \mathbf{V}\times\mathbf{B}) - d_t\mathbf{V}$ and $\mathsf{W} = \nabla\mathbf{V} + (\nabla\mathbf{V})^{\mathrm{T}} - \frac{2}{3}\nabla\cdot\mathbf{V}\mathsf{I}$

Closures: express $\mathbf{h}_a(\mathsf{n}^{11}_a)$, $\boldsymbol{\pi}_a(\mathsf{n}^{20}_a)$, Q_a, $\mathbf{R}_a$ in terms of $n_a, \mathbf{V}_a, T_a$

$$\mathbf{h}_{\mathrm{e}} = -\kappa^{\mathrm{e}}_{\parallel}\nabla_{\parallel}T_{\mathrm{e}} - \kappa^{\mathrm{e}}_{\perp}\nabla_{\perp}T_{\mathrm{e}} - \kappa^{\mathrm{e}}_{\times}\nabla_{\times}T_{\mathrm{e}} + \beta_{\parallel}T_{\mathrm{e}}\mathbf{V}_{\mathrm{ei}\parallel} + \beta_{\perp}T_{\mathrm{e}}\mathbf{V}_{\mathrm{ei}\perp} + \beta_{\times}T_{\mathrm{e}}\mathbf{V}_{\mathrm{ei}\times}$$

$$\mathbf{R}_{\mathrm{e}} = -\alpha_{\parallel}\mathbf{V}_{\mathrm{ei}\parallel} - \alpha_{\perp}\mathbf{V}_{\mathrm{ei}\perp} + \alpha_{\times}\mathbf{V}_{\mathrm{ei}\times} - \beta_{\parallel}\nabla_{\parallel}T_{\mathrm{e}} - \beta_{\perp}\nabla_{\perp}T_{\mathrm{e}} - \beta_{\times}\nabla_{\times}T_{\mathrm{e}}$$

General moment expansion [Ji and Held 2006 2008 2009]
Irreducible Hermite polynomials [Chapman 1916 Enskog 1917 Grad 1963]

- Moment expansion: m_a^{lk} (vs. M_a^{lk}) are symmetric traceless fluid moments

$$f_a(t,\mathbf{r},\mathbf{v}) = f_a^{\mathrm{M}} \sum_{lk} \mathsf{m}_a^{lk}(t,\mathbf{r}) \cdot \hat{\mathsf{p}}_a^{lk} \quad \text{vs.} \quad f_a(t,\mathbf{r},\mathbf{v}) = f_a^{\mathrm{m}} \sum_{lk} \mathsf{M}_a^{lk}(t,\mathbf{x}) \cdot \hat{\mathsf{P}}_a^{lk}$$

$$\mathsf{n}_a^{lk}(t,\mathbf{r}) \equiv n_a \mathsf{m}_a^{lk} = \int d\mathbf{v} \hat{\mathsf{p}}_a^{lk} f_a \qquad \mathsf{N}_a^{lk}(t,\mathbf{x}) \equiv n_a \mathsf{M}_a^{lk} = \int d\mathbf{v} \hat{\mathsf{P}}_a^{lk} f_a$$

- $\hat{\mathsf{p}}^{lk}$'s are orthonormal, irreducible, tensorial polynomials and form a complete set

$$\int d\mathbf{v} \hat{\mathsf{p}}^{jp} \hat{\mathsf{p}}^{lk} \cdot \mathsf{m}^{lk} f^{\mathrm{M}} = \delta_{jl}\delta_{pk}\mathsf{n}^{jp}, \quad \hat{\mathsf{p}}^{lk} = \frac{1}{\sqrt{\sigma_{lk}}}\mathsf{p}^{lk}$$

$$\begin{aligned} \mathsf{p}_a^{lk} &= \mathsf{P}^l(\mathbf{c}_a) L_k^{(l+1/2)}(c_a^2) \quad \text{vs.} \quad \mathsf{P}_a^{lk} = \mathsf{P}^l(\mathbf{s}_a) L_k^{(l+1/2)}(s_a^2) \\ &= \text{(normalization)(harmonic tensor)(associated Laguerre polynomial)} \end{aligned}$$

$$f_a^{\mathrm{M}} = \frac{n_a}{\pi^{3/2} v_{Ta}^3} e^{-c_a^2} \text{ (Maxwellian distribution)} \quad \text{vs.} \quad f_a^{\mathrm{m}} = \frac{n_a}{\pi^{3/2} v_{Ta}^3} e^{-s_a^2}$$

$$\mathbf{c}_a = \frac{\mathbf{w}_a}{v_{Ta}} = \frac{\mathbf{v} - \mathbf{V}_a}{v_{Ta}}, \quad v_{Ta} = \sqrt{\frac{2T_a}{m_a}}, \quad \text{vs.} \quad \mathbf{s}_a = \frac{\mathbf{v}}{v_{Ta}}$$

Moments of the Landau collision operator

$$\frac{\gamma_{ab}}{2m_a}\int d\mathbf{v}\mathsf{P}^{jp}(\mathbf{c}_a)\frac{\partial}{\partial\mathbf{v}}\cdot\left[\frac{\partial}{\partial\mathbf{v}}\cdot\left(f_a\frac{\partial^2 G_{+,b}}{\partial\mathbf{v}\partial\mathbf{v}}\right)-2\left(1+\frac{m_a}{m_b}\right)\frac{\partial}{\partial\mathbf{v}}\cdot\left(f_a\frac{\partial G_{-,b}}{\partial\mathbf{v}}\right)\right]$$

Rosenbluth potential $G_{\pm,b}(\mathbf{v})=\int d\mathbf{v}' f_b(\mathbf{v}')|\mathbf{v}-\mathbf{v}'|^{\pm 1}$

- Exact collisional moments in the total velocity expansion [Ji and Held 2006]

$$\int d\mathbf{v}\mathsf{P}^{jp}\left(\frac{\mathbf{v}}{v_{Ta}}\right)C\left(f_a^{\mathrm{M}}\mathsf{M}_a^{lk}\cdot\mathsf{P}^{lk}\left(\frac{\mathbf{v}}{v_{Ta}}\right),f_b^{\mathrm{M}}\mathsf{M}_b^{nq}\cdot\mathsf{P}^{nq}\left(\frac{\mathbf{v}}{v_{Tb}}\right)\right)$$

- Small mass ratio approximation in the random velocity expansion [Ji and Held 2009]

$$\int d\mathbf{v}\mathsf{P}^{jp}(\mathbf{c}_{\mathrm{e}})C\left(f_{\mathrm{e}}^{\mathrm{M}}\mathsf{m}_{\mathrm{e}}^{lk}\cdot\mathsf{P}^{lk}(\mathbf{c}_{\mathrm{e}}),f_{\mathrm{i}}^{\mathrm{M}}\right) = \sigma_j A_{\mathrm{ei}}^{jp,lk}\mathsf{m}_{\mathrm{e}}^{lk}$$

$$\int d\mathbf{v}\mathsf{P}^{jp}(\mathbf{c}_{\mathrm{i}})C\left(f_{\mathrm{e}}^{\mathrm{M}},f_{\mathrm{i}}^{\mathrm{M}}\mathsf{m}_{\mathrm{i}}^{nq}\cdot\mathsf{P}^{nq}(\mathbf{c}_{\mathrm{i}})\right) = \sigma_j B_{\mathrm{ei}}^{jp,lk}\mathsf{m}_{\mathrm{i}}^{lk}$$

  - Assumed $v_{T\mathrm{i}}\ll v_{T\mathrm{e}}$ and $|\mathbf{V}_{\mathrm{e}}-\mathbf{V}_{\mathrm{i}}|\ll v_{T\mathrm{e}}$: not valid for runaway electrons

- Need to calculate the collisional moments for arbitrary relative velocity

$$\int d\mathbf{v}\mathsf{P}^{jp}\left(\frac{\mathbf{v}-\mathbf{V}_a}{v_{Ta}}\right)C\left(f_a^{\mathrm{M}}\mathsf{m}_a^{lk}\cdot\mathsf{P}^{lk}\left(\frac{\mathbf{v}-\mathbf{V}_a}{v_{Ta}}\right),f_b^{\mathrm{M}}\mathsf{m}_b^{nq}\cdot\mathsf{P}^{nq}\left(\frac{\mathbf{v}-\mathbf{V}_b}{v_{Tb}}\right)\right)$$

How to calculate exact collisional moments for arbitrary relative velocity

- Change derivatives, $\dfrac{\partial G_\pm(\mathbf{v}-\mathbf{V}_b)}{\partial \mathbf{v}} = -\dfrac{\partial G_\pm}{\partial \mathbf{V}_b}$
- Integrate by parts to differentiate each term of polynomial $\mathsf{P}^{jp}(\mathbf{c}_a)$
- Change velocity variables [Schunk and Nagy 2000]

$$\begin{matrix} \mathbf{w}_\star = X_{ba}\mathbf{w}_a + X_{ab}\mathbf{w}'_b \\ \mathbf{w}_\dagger = \mathbf{w}_a - \mathbf{w}'_b \end{matrix} \quad \leftrightarrow \quad \begin{matrix} \mathbf{w}_a = v_\star(\mathbf{c}_* + \chi_a \mathbf{c}_\dagger) \\ \mathbf{w}'_b = v_\star(\mathbf{c}_\star + \chi_b \mathbf{c}_\dagger) \end{matrix} \; , \qquad \begin{matrix} v_\star^{-2} = v_{Ta}^{-2} + v_{Tb}^{-2} \\ v_\dagger^2 = v_{Ta}^2 + v_{Tb}^2 \end{matrix}$$

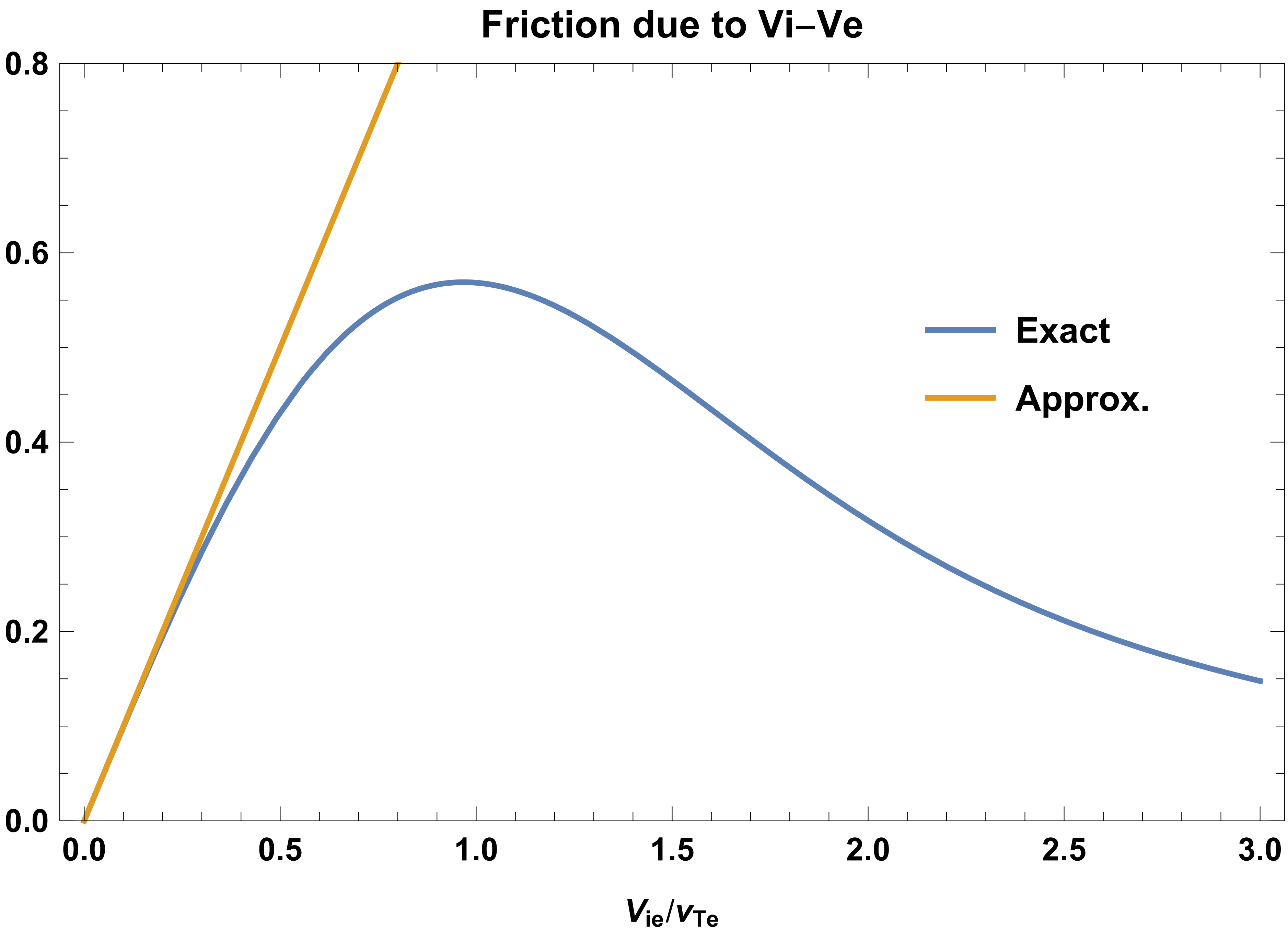
where $X_{ab} = (1 + v_{Ta}^2/v_{Tb}^2)^{-1}$, $X_{ba} = 1 - X_{ab}$, $\chi_a = v_{Ta}/v_{Tb}$, $\chi_b = -v_{Tb}/v_{Ta}$

$$\int d\mathbf{v} \int d\mathbf{v}' f_a^{\mathrm{M}} f_b^{\mathrm{M}} |\mathbf{v}-\mathbf{v}'|^\pm = n_a n_b v_\dagger^{\pm 1} \int d\mathbf{c}_\star \int d\mathbf{c}_\dagger \frac{e^{-c_*^2}}{\pi^{3/2}} \frac{e^{-c_\dagger^2}}{\pi^{3/2}} |\mathbf{c}_\dagger + \mathbf{x}|^\pm$$

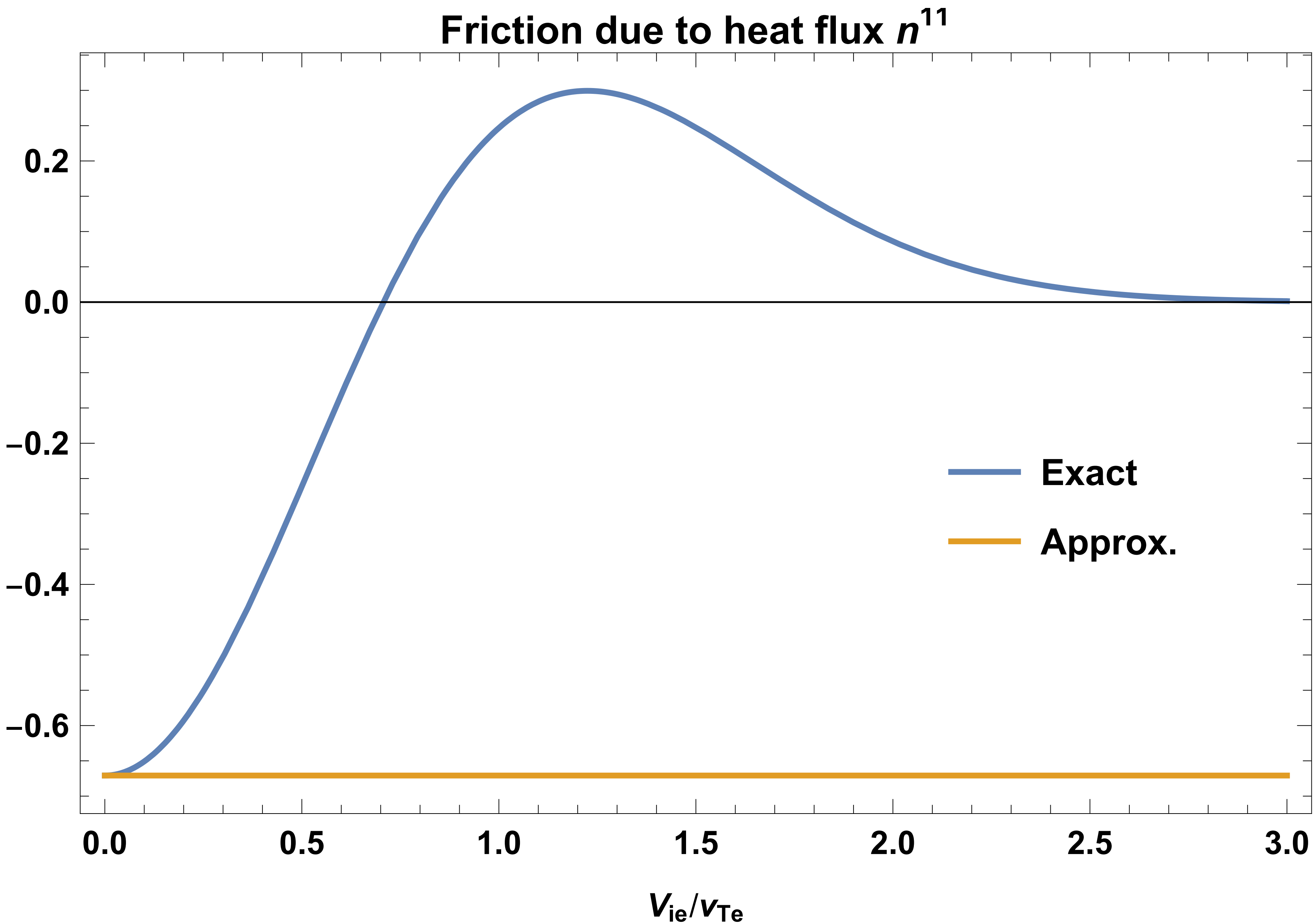
where $\mathbf{x} = (\mathbf{V}_b - \mathbf{V}_a)/v_\dagger$

- Expand all $\mathbf{c}_a$ and $\mathbf{c}_b$ variables: involve multiple summations
- Perform $\mathbf{c}_\star$ integration using $\displaystyle\int d\mathbf{c}_\star \mathbf{c}_\star^n c^{2u} \frac{e^{-c_\star^2}}{\pi^{3/2}} = \frac{2[u+(n+1)/2]!}{\pi^{1/2}(n+1)} \{\mathsf{I}^{n/2}\}$
- Simplify all inner-products and symmetrizations
- Perform $\mathbf{c}_\dagger$ integration using the Rosenbluth potentials in Ji and Held 2006
- Differentiate with respect to $\mathbf{V}_b$ or $\mathbf{x}$
- Final results involve only algebraic summations of $G_\pm^{np}(x)\left\{\mathbf{x}^j \cdot^r \mathsf{m}^{lk} \mathsf{I}^i\right\}$ terms

Friction responding to $\mathrm{V_i - V_e}$



Friction responding to heat flux (n^{11})



Work to be done soon

- Find bugs
  - All explicit formulas are written in Mathematica code
- Replace the collision matrices and calculate closures (straightforward)
  - High collisionality
    - Modify Braginskii closures
  - Low (general) collisionality
    - Modify integral closures